Engineering Maths First Aid Kit

2.24

Partial Fractions 2

1. Worked Examples

Example

Express $\dfrac{5x-4}{x^2-x-2}$ as the sum of its partial fractions.

Solution

First we factorise the denominator: $x^2 - x - 2 = (x+1)(x-2)$. Next, examine the form of the factors. The factor $(x+1)$ is a linear factor and produces a partial fraction of the form $\frac{A}{x+1}$. The factor $(x-2)$ is also a linear factor, and produces a partial fraction of the form $\frac{B}{x-2}$. Hence

$$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \qquad (1)$$

where A and B are constants which must be found. Finally we find the constants. Writing the right-hand side using a common denominator we have

$$\frac{5x-4}{(x+1)(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

The denominators on both sides are the same, and so the numerators on both sides must be the same too. Thus

$$5x - 4 = A(x-2) + B(x+1) \qquad (2)$$

We shall first demonstrate how to find A and B by **substituting specific values for** x. By appropriate choice of the value for x, the right-hand side of Equation 2 can be simplified. For example, letting $x = 2$ we find $6 = A(0) + B(3)$, so that $6 = 3B$, that is $B = 2$. Then by letting $x = -1$ in Equation 2 we find $-9 = A(-3) + B(0)$, from which $-3A = -9$, so that $A = 3$. Substituting these values for A and B into Equation 1 gives

$$\frac{5x-4}{x^2-x-2} = \frac{3}{x+1} + \frac{2}{x-2}$$

The constants can also be found by **equating coefficients**. From Equation 2 we have

$$\begin{aligned} 5x - 4 &= A(x-2) + B(x+1) \\ &= Ax - 2A + Bx + B \\ &= (A+B)x + B - 2A \end{aligned}$$

Comparing the coefficients of x on the left- and right-hand sides gives $5 = A + B$. Comparing the constant terms gives $-4 = B - 2A$. These simultaneous equations in A and B can be solved to find $A = 3$ and $B = 2$ as before. Often a combination of the two methods is needed.

Example

Express $\dfrac{2x^2+3}{(x+2)(x+1)^2}$ in partial fractions.

Solution

The denominator is already factorised. Note that there is a linear factor $(x+2)$ and a repeated linear factor $(x+1)^2$. So we can write

$$\frac{2x^2+3}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \tag{3}$$

The right hand side is now written over a common denominator to give

$$\frac{2x^2+3}{(x+2)(x+1)^2} = \frac{A(x+1)^2 + B(x+2)(x+1) + C(x+2)}{(x+2)(x+1)^2}$$

Therefore

$$2x^2 + 3 = A(x+1)^2 + B(x+2)(x+1) + C(x+2) \tag{4}$$

A and C can be found by substituting values for x which simplify the right-hand side. For example if $x = -1$ we find $2(-1)^2 + 3 = A(0) + B(0) + C$ from which $C = 5$. Similarly if we choose $x = -2$ we find $8 + 3 = A(-1)^2 + B(0) + C(0)$ so that $A = 11$. To find B we shall use the method of **equating coefficients**, although we could equally have substituted any other value for x. To equate coefficients we remove the brackets on the right-hand side of Equation 4. After collecting like terms we find that Equation 4 can be written

$$2x^2 + 3 = (A+B)x^2 + (2A+3B+C)x + (A+2B+2C)$$

By comparing the coefficients of x^2 on both sides we see that $(A+B)$ must equal 2. Since we already know $A = 11$, this means $B = -9$. Finally substituting our values of A, B and C into Equation 3 we have $\dfrac{2x^2+3}{(x+2)(x+1)^2} = \dfrac{11}{x+2} - \dfrac{9}{x+1} + \dfrac{5}{(x+1)^2}$.

Exercises

1. Show that $\frac{x-1}{6x^2+5x+1} = \frac{3}{2x+1} - \frac{4}{3x+1}$.

2. Show that $\frac{s+4}{s^2+s} = \frac{4}{s} - \frac{3}{s+1}$.

3. The fraction $\dfrac{5x^2+4x+11}{(x^2+x+4)(x+1)}$ has a quadratic factor in the denominator which cannot be factorised. Thus the required form of the partial fractions is

$$\frac{5x^2+4x+11}{(x^2+x+4)(x+1)} = \frac{Ax+B}{x^2+x+4} + \frac{C}{x+1}$$

Show that $\dfrac{5x^2+4x+11}{(x^2+x+4)(x+1)} = \dfrac{2x-1}{x^2+x+4} + \dfrac{3}{x+1}$.